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Second-class motions of a shallow liquid

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When a basin containing a shallow liquid is rotating with angular velocity Ω and the dimensionless number

$$\epsilon = 4\Omega^2 L^2/gM$$

is small (L and M are typical horizontal and vertical dimensions), then, to a first approximation, the second-class motions behave as if the free surface of the liquid were fixed in its equilibrium position. The lower second-class modes of such a liquid, contained in a paraboloid, are relatively easy to describe on the basis of this approximation. When the liquid is rotating within an elliptical paraboloid and the sense of rotation is opposite to that of the container itself, the motion is unstable for a range of small angular velocities. Such unstable motions always exert a couple tending to oppose the rotation of the container.

1. Introduction

Shallow-water motions in a non-rotating system can conveniently be divided into two classes. In simple circumstances these classes are quite distinct; the first consists of long gravity waves, or oscillations, with typical frequencies of order ν, where

$$\nu^2 = Mg/L^2, \tag{1.1}$$

M and L being characteristic vertical and horizontal scales respectively; the second consists of steady rotational motions (on a linear theory). The first-class motions involve a significant disturbance of the free surface, the essence of the motion being the alternation between kinetic and potential energies. The second-class motions do not significantly disturb the free surface and the energy is almost entirely kinetic; the motion is essentially rotational.

When the whole system rotates with angular velocity Ω, the second-class motions may become oscillatory with frequency of order Ω (see Lamb 1932, §§206, 212 and 223). If we let the relative vorticity be ζ (i.e. the vorticity relative to axes rotating with the system) then the absolute vorticity is $\zeta+2\Omega$ and the potential vorticity is given by $(\zeta+2\Omega)/h$ where h is the depth of the liquid. This quantity is important in shallow-water theory because it is conserved, that is it satisfies

$$D[(\zeta+2\Omega)/h]/Dt = 0. \tag{1.2}$$

If in the static undisturbed state the potential vorticity is constant then the linear approximation to second-class motions will be steady though the flow

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will be in 'geostrophic' equilibrium and the free surface will no longer be undisturbed. However, when there is a gradient of potential vorticity in the undisturbed state the second-class motions will generally be oscillatory, though there may be particular modes that are steady. In the present investigation the gradient of potential vorticity is provided by the variation of the equilibrium depth, though in the well-known simple example of oscillatory second-class motions, Rossby waves (Rossby 1939), the gradient of potential vorticity is provided by the variation in Ω consequent on the sphericity of the earth.

Although the existence of second-class motions has been known for some time, it is only in recent years that they have received much attention. This recent work has been largely stimulated by problems associated with numerical weather forecasting by dynamical processes, by the increasing number of observations of oceanic motions and by the realization that the important large-scale motions of the atmosphere and the current systems of the oceans are essentially of the second class. There are still very few studies of simple systems by which one can hope to gain insight into the behaviour of second-class motions and these studies have been concerned with the effect of the sphericity of the earth on the motion of a liquid of constant depth (see Goldsbrough 1933; Arons & Stommel 1956; Longuet-Higgins 1964, 1965).

It is the purpose of this paper to examine the behaviour of some simple examples of second-class motions, both stable and unstable, of a slowly rotating system, the gradient of potential vorticity being provided by variation in equilibrium depth. It is first shown that the upper surface of the liquid may be regarded as fixed,† provided the dimensionless number ϵ is small where

$$\epsilon = 4\Omega^2L^2/gM. \tag{1.3}$$

This is equivalent to the parameter employed by Longuet-Higgins (1965), where L is the radius of the earth and M the liquid depth. (ϵ also becomes the reciprocal of the 'Burger number' of stratified flow if g is replaced by $g\Delta\rho/\rho$, see Burger 1958 and Phillips 1963.) With this simplification it is relatively easy to examine the behaviour of the lower second-class modes in an elliptic paraboloid and to show that simple 'elliptic' rotation in such a container is *unstable* for certain angular velocities.

2. Formulation

Let the shape of a basin containing a *shallow* liquid be described by the equation

$$Z = Z(X, Y). \tag{2.1}$$

Capital letters are used here for the various variables; small letters are used for the corresponding dimensionless variables defined in the following section. If the basin is rotating with constant angular velocity Ω, the motion of the liquid is governed by the three equations

$$DU/DT + g\,\partial(H+Z-\tfrac{1}{2}\Omega^2R^2/g)/\partial X = 2\Omega V, \tag{2.2}$$

$$DV/DT + g\,\partial(H+Z-\tfrac{1}{2}\Omega^2R^2/g)/\partial Y = -2\Omega U, \tag{2.3}$$

$$\partial H/\partial T + \partial(HU)/\partial X + \partial(HV)/\partial Y = 0, \tag{2.4}$$

† A similar result has recently been obtained by Phillips (1965).

where H is the depth of the liquid and $R^2 = X^2 + Y^2$. There are two types of boundary conditions that usually occur. If the basin has a vertical periphery, then the boundary of the liquid is fixed and the component of the velocity normal to the boundary must vanish at the boundary. If the basin is sloping at the edge of the liquid (like a beach) then the liquid boundary is free and we have $H = 0$, implying that $DH/DT = 0$, akin to the condition of constant pressure at a free surface. We will show later that these two types of boundary condition become identical for second-class motions of a slowly rotating system.

Before discussing equations (2.2)–(2.4) in more general terms, we notice that when the equations are linearized in the usual way for small motion and when the boundaries are free, there is always a *steady* solution of the form

$$H = F(Z^*) - Z^*, \tag{2.5}$$

$$2\Omega U = -g\,\partial F/\partial Y, \tag{2.6}$$

$$2\Omega V = g\,\partial F/\partial X, \tag{2.7}$$

$$Z^* = Z - \tfrac{1}{2}\Omega^2 R^2/g, \tag{2.8}$$

and F is an arbitrary function. In this steady motion the streamlines are parallel to the contours of Z^*. This solution gives only a subclass of the second-class motions and some special cases of it are discussed in § 4 (see also Lamb 1932, § 207).

We also notice that when Z^* is constant and the boundaries are fixed, there is a steady solution of the linearized equations of the form

$$2\Omega U = -g\,\partial H/\partial Y, \tag{2.9}$$

$$2\Omega V = g\,\partial H/\partial X, \tag{2.10}$$

where H is constant on the boundary but otherwise arbitrary. This solution gives all possible second-class motions of this particular system, the liquid being in 'geostrophic' equilibrium and the velocity field being non-divergent.

When the basin is fixed to the rotating earth and the vertical co-ordinate is taken perpendicular to the geopotential surfaces, the centripetal terms, involving $\tfrac{1}{2}\Omega^2 R^2$, no longer appear in equations (2.2) and (2.3) (having been absorbed in $\mathfrak{g}$), Z^* becomes equal to Z and the above condition, for the degeneration of the second-class motions into steady motions, becomes merely $Z = \text{const}$. This is perhaps one reason why second-class motions have not received much attention, most theoretical work has been concerned with basins of constant depth on a rotating earth in which oscillatory second-class motions cannot occur unless the basin is large enough for the sphericity of the earth to be significant.

3. Second-class motions of a slowly rotating system

Let us now choose a horizontal scale L and a vertical scale M and define dimensionless quantities u, v, x, y, t, h, z by

$$\left.\begin{aligned} U &= 2L\Omega u, & V &= 2L\Omega v, \\ X &= Lx, & Y &= Ly, \\ H &= Mh, & Z &= Mz, \\ \text{and} \qquad & & T &= t/2\Omega. \end{aligned}\right\} \tag{3.1}$$

Equations (2.2)–(2.4) then become

$$\epsilon(Du/Dt - v - \tfrac{1}{4}x) + \partial(h+z)/\partial x = 0, \tag{3.2}$$

$$\epsilon(Dv/Dt + u - \tfrac{1}{4}y) - \partial(h+z)/\partial y = 0, \tag{3.3}$$

$$\partial h/\partial t + \partial(hu)/\partial x + \partial(hv)/\partial y = 0, \tag{3.4}$$

where ϵ is the dimensionless number defined by (1.3). We now assume, first, that the motion has time and space scales appropriate to the transformation (3.1), and secondly that the rotation is slow in the sense that ϵ is *small*. We are thereby confining our attention to second-class motions, that is, motions which have a characteristic frequency of order Ω. First-class motions have frequencies of order ν_1, say, where

$$(\nu_1/2\Omega)^2 = Mg/(4\Omega^2L^2) = \epsilon^{-1}, \tag{3.5}$$

which is, under these assumptions, a higher-order of magnitude than the frequency of second-class motions.

The condition for the validity of the shallow-water approximation is almost automatically satisfied by this type of motion. To see this we first notice that equations (3.2) and (3.3) imply that the slope of the free surface $(h+z)$ is small, which indicates that the vertical accelerations will be greatest at the base of the liquid where it runs up or down the slope of the basin. We therefore have

$$W = U\,\partial Z/\partial X + V\,\partial Z/\partial Y = 2\Omega M(u\,\partial z/\partial x + v\,\partial z/\partial y).$$

The vertical velocity is of order $2\Omega M$ and the vertical acceleration is of order $4\Omega^2 M$. The conditions for the validity of the shallow-water approximation, that the vertical acceleration should be small by comparison with g, can now be written

$$4\Omega^2 M/g \ll 1,$$

or

$$\epsilon(M/L)^2 \ll 1,$$

and we see that this is satisfied if ϵ is small even when M and L are of the same order of magnitude.

We now seek first approximations to the basic equations (3.2)–(3.4) for small values of the parameter ϵ. We do this by expanding the dependent variables in a series of ascending powers of ϵ and then obtain a series of equations by equating coefficients of ϵ when these expansions are substituted into equations (3.2)–(3.4). Let us put

$$\left.\begin{aligned} u &= u_0 + u_1\epsilon + \ldots + u_n\epsilon^n \ldots, \\ v &= v_0 + v_1\epsilon + \ldots + v_n\epsilon^n \ldots, \\ h &= h_0 + h_1\epsilon + \ldots + h_n\epsilon^n \ldots. \end{aligned}\right\} \tag{3.6}$$

The terms of zero order in ϵ then give

$$\partial(h_0+z)/\partial x = 0, \tag{3.7}$$

$$\partial(h_0+z)/\partial y = 0, \tag{3.8}$$

$$\partial h_0/\partial t + \partial(h_0u_0)/\partial x + \partial(h_0v_0)/\partial y = 0. \tag{3.9}$$

Equations (3.7) and (3.8) show that, to a first approximation, the free surface is *level*; we also know that, if volume is to be conserved, this level must coincide with that of the undisturbed free surface. If we measure z from the level of the undisturbed free surface then we have

$$h_0 + z = 0, \tag{3.10}$$

and equation (3.9) becomes

$$\partial(zu_0)/\partial x + \partial(zv_0)/\partial y = 0. \tag{3.11}$$

The terms of order ϵ in equations (3.2) and (3.3) give

$$D_0 u_0/D_0 t - v_0 - \tfrac{1}{4}x + \partial h_1/\partial x = 0, \tag{3.12}$$

and

$$D_0 v_0/D_0 t + u_0 - \tfrac{1}{4}y + \partial h_1/\partial y = 0, \tag{3.13}$$

which together with (3.11) form a determinate set of equations in the three unknowns u_0, v_0 and h_1. The operator $D_0/D_0 t$ is defined by

$$D_0/D_0 t \equiv \partial/\partial t + u_0 \partial/\partial x + v_0 \partial/\partial y. \tag{3.14}$$

At a free boundary Dh/Dt is zero and, when this is expanded in powers of ϵ, the terms of zero order give

$$u_0 \partial z/\partial x + v_0 \partial z/\partial y = 0, \tag{3.15}$$

which shows that the velocity is parallel to the contour of z and therefore parallel to the boundary since this is itself a contour of z ($z = 0$). The boundary conditions at free and fixed boundaries are therefore identical (what was previously a free boundary has now become essentially a fixed boundary with zero depth).

The set of equations (3.11)–(3.13) is exactly the same as the set which governs the motion of a shallow liquid in a basin of shape given by (2.1), but with a rigid upper surface, the pressure exerted by the liquid on this surface being proportional to h_1. On the other hand, from the point of view of (3.6) we must regard $h_1\epsilon$ as the small deviation of the free surface from its equilibrium position.

Perhaps the most convenient way to express these equations in terms of a single dependent variable is first to eliminate h_1 by determining a vorticity equation from (2.16) and (2.17) thus

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + 1\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{3.16}$$

and then to express (3.16) in terms of a 'stream function' by putting

$$zu = -\partial(z^2\psi)/\partial y, \quad zv = \partial(z^2\psi)/\partial x,$$

so

$$u = -2\psi\, \partial z/\partial y - z\, \partial\psi/\partial y, \quad v = 2\psi\, \partial z/\partial x + z\, \partial\psi/\partial x, \tag{3.17}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial z}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial z}{\partial y}, \tag{3.18}$$

and

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\psi\nabla^2 z + 3\nabla\psi\,.\,\nabla z + z\nabla^2\psi. \tag{3.19}$$

The suffix zero has now been omitted as no longer relevant. Equation (3.11) is now automatically satisfied and so also is the boundary condition at 'free' boundaries (where $z = 0$) provided $\nabla\psi$ remains finite; this is not a necessary condition but is sufficient for present purposes.

4. Second-class oscillations in an elliptic paraboloid

We now apply the preceding ideas to determine the second-class motions in an elliptic paraboloid, such a basin being the simplest that has both the necessary variation in depth and a low enough degree of symmetry to avoid degenerate modes. Suppose the liquid has central depth M (chosen equal to our vertical 'scale') and take the origin of co-ordinates at the centre of the liquid surface, then (2.1) becomes

$$Z = \tfrac{1}{2}\alpha X^2 + \tfrac{1}{2}\beta Y^2 - M. \tag{4.1}$$

Let us take L to be a typical horizontal dimension of the liquid defined by

$$L^2 = 2M/(\alpha+\beta), \tag{4.2}$$

then, using the dimensionless variables defined by (3.1), (4.1) reduces to

$$z = \tfrac{1}{2}[(1-a)x^2 + (1+a)y^2] - 1, \tag{4.3}$$

where

$$a = (\beta-\alpha)/(\beta+\alpha) \tag{4.4}$$

(a is supposed positive for definiteness, the major axis of the 'free' surface then being in the x-direction). Equations (3.17) can now be written

$$\left.\begin{aligned} u &= -2(1+a)y\psi - z\,\partial\psi/\partial y \\ v &= +2(1-a)x\psi + z\,\partial\psi/\partial x. \end{aligned}\right\} \tag{4.5}$$

and

We linearize the vorticity equation (3.16) in the usual way for small motion by neglecting the terms of the second degree in the velocity and substitute for u and v using (4.5) to obtain

$$4\frac{\partial\psi}{\partial t} + 3\left[(1-a)\,x\frac{\partial^2\psi}{\partial x\,\partial t} + (1+a)\,y\frac{\partial^2\psi}{\partial y\,\partial t}\right] + z\nabla^2\left(\frac{\partial\psi}{\partial t}\right) + (1-a)\,x\frac{\partial\psi}{\partial y} - (1+a)\,y\frac{\partial\psi}{\partial x} = 0. \tag{4.6}$$

We have now reduced the problem of determining the character of small second-class motions in a slowly rotating elliptic paraboloid, to the problem of finding the character of the solutions of the single equation (4.6). This equation has polynomial solutions and it seems likely that these solutions are the only ones for which ψ is finite throughout the region of interest (i.e. $z \leqslant 0$). The simplest non-trivial solution is

$$\psi = \psi_{00}\ \text{(constant)}, \tag{4.7}$$

and

$$u = -2(1+a)y\psi_{00}, \quad v = 2(1-a)x\psi_{00}. \tag{4.8}$$

This is *elliptic rotation* with angular velocity ω where

$$\omega = -2\psi_{00}. \tag{4.9}$$

Each liquid column moves in an ellipse that is similar to the elliptical periphery of the liquid. The range of potential vorticities (dimensionless) in the undisturbed state is $(1, \infty)$, whereas in a state of elliptic rotation the range is $[(1+4\psi_{00}), \infty]$, so that elliptic rotation can only be generated by processes that create or destroy potential vorticity (the same is true of the other modes discussed in this section).

We next consider the case where ψ is a first-degree polynomial in x and y, the coefficients being functions of time, thus

$$\psi = \psi_{10}x + \psi_{01}y. \tag{4.10}$$

We need only consider polynomials in which the terms all have degrees of the same parity, because equation (4.6) only provides relationships between the coefficients of terms of the same parity. On substitution into equation (4.6) we find

$$\{(7-3a)\,d\psi_{10}/dt\} + (1-a)\,\psi_{01} = 0, \tag{4.11}$$

$$\{(7+3a)\,d\psi_{01}/dt\} - (1+a)\,\psi_{10} = 0. \tag{4.12}$$

The motion is therefore oscillatory with frequency ν where

$$\nu^2 = (1-a^2)/(49-9a^2). \tag{4.13}$$

The exact value of this frequency (i.e. without restriction on ϵ) is given by Ball (1965, equation (9.1)) for the case of a basin rotating with the earth. The frequency appears as the smallest root of a fifth-degree equation, which when put in appropriate dimensionless form is easily seen to reduce to the value (4.13) for small ϵ. The other roots of this equation represent the frequencies of modes of the first class. In the case of a circular paraboloid $a = 0$ and (4.13) gives $\nu = \frac{1}{7}$, in agreement with Miles & Ball (1963), equations (3.16*b*), $s = 1, j = 2$.

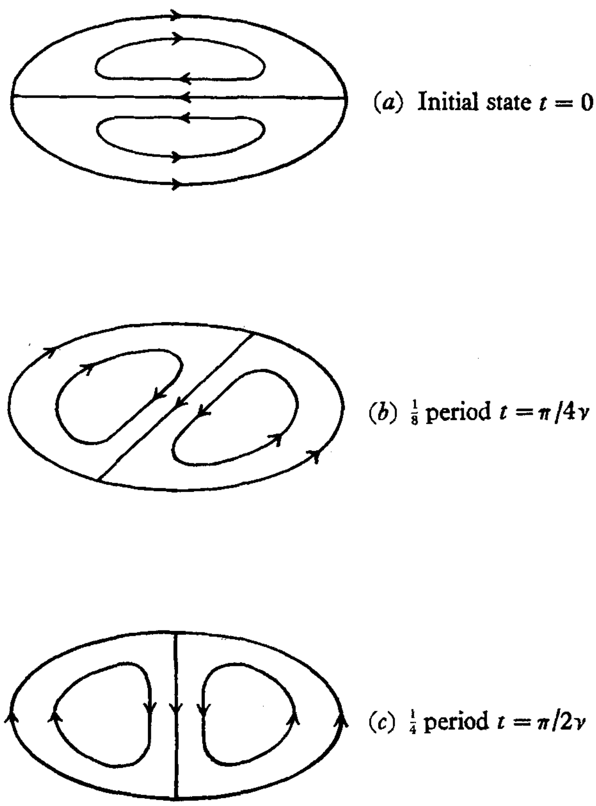


FIGURE 1. Streamlines for the 'linear mode'.

The solution can be written

$$\psi = A[x\{(1-a)(7+3a)\}^{\frac{1}{2}}\cos(\nu t)+y\{(1+a)(7-3a)\}^{\frac{1}{2}}\sin(\nu t)], \qquad (4.14)$$

and the streamlines consist of two opposite circulation cells separated by the diameter $\psi = 0$. This diameter rotates in a *positive* direction (i.e. in the same direction as the direction of rotation of the system) see figure 1. The actual motion of the liquid particles is quite different from that suggested by the streamlines. All particles move in small ellipses of various eccentricities and orientations. Those near the principal axes of the container move in ellipses whose principal axes are parallel to those of the container, whereas those near the boundary of the liquid execute simple harmonic oscillations parallel to the boundary.

We next consider the modes for which ψ is a quadratic function of x and y,

$$\psi = \psi_{00}+\psi_{20}x^2+2\psi_{11}xy+\psi_{02}y^2, \qquad (4.15)$$

which, on substitution into (4.6), leads to

$$\left.\begin{aligned}\{(11-7a)\,d\psi_{20}/dt\}+\{(1-a)\,d\psi_{02}/dt\}+2(1-a)\,\psi_{11} &= 0,\\ \{10\,d\psi_{11}/dt\}+(1-a)\,\psi_{02}-(1+a)\,\psi_{20} &= 0,\\ \{(11+7a)\,d\psi_{02}/dt\}+\{(1+a)\,d\psi_{20}/dt\}-2(1+a)\,\psi_{11} &= 0,\end{aligned}\right\} \qquad (4.16)$$

and

$$d(2\psi_{00}-\psi_{20}-\psi_{02})/dt = 0. \qquad (4.17)$$

Equation (4.17) enables ψ_{00} to be determined when equations (4.16) have been solved. The frequency equation for (4.16) is

$$\nu[5\nu^2(5-2a^2)-(1-a^2)] = 0, \qquad (4.18)$$

which shows that there is a steady mode ($\nu = 0$) and a mode with frequency given by

$$\nu^2 = (1-a^2)/[5(5-2a^2)]. \qquad (4.19)$$

In the case of a circular paraboloid $a = 0$ and (4.19) gives $\nu = \frac{1}{5}$ in agreement with Miles & Ball (1963), equation (3.16*b*), $s = 2, j = 2$.

The simplest form of the steady-state solution is

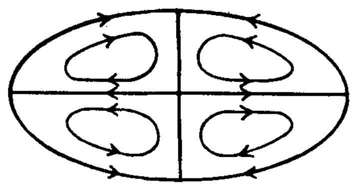
$$\psi = A[(1-a)x^2+(1+a)y^2], \qquad (4.20)$$

corresponding to a central negatively rotating region surrounded by an annular positively rotating region (assuming $A > 0$). As in the case of elliptic rotation the liquid particles move in ellipses along the contours of z. Both this solution and elliptic rotation (4.9) are special cases of the approximate form of the steady solution (2.5)–(2.7). Whenever we assume that ψ is a polynomial of even order, $2n$ say, we must obtain a steady solution in which ψ is an nth-degree polynomial function of z (corresponding to F, in equations (2.5)–(2.7), being an $(n+1)$th-degree polynomial).

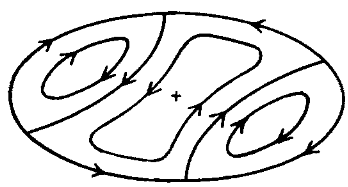
The oscillatory solution is given by

$$\psi = A[\{(1+a)(3-2a)y^2-(1-a)(3+2a)x^2+a\}\sin\nu t+\{6(1-a^2)/5\nu\}xy\cos\nu t]. \qquad (4.21)$$

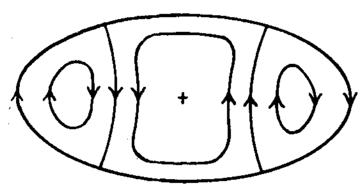
When $t = 0$ the streamlines form four circulation cells separated by the principal axes of the elliptical container, the sense of the circulation having the same sign as xy ($A > 0$). The two positive circulation cells merge and the pattern changes so that in the quarter-wave state ($t = \pi/2\nu$) there are three cells, a positive one in the middle and a negative one at each end of the ellipse, the cells being separated by hyperbolic arcs (see figure 2). The half-wave state is similar to the initial state except that the circulations are of opposite sense. As in the oscillatory mode considered previously, these circulation cells are only apparent, the liquid particles in fact move in small ellipses.



(*a*) Initial state $t = 0$



(*b*) $\frac{1}{8}$ period $t = \pi/4\nu$



(*c*) $\frac{1}{4}$ period $t = \pi/2\nu$

FIGURE 2. Streamlines for the 'quadratic mode'.

One can continue the sequence and assume that ψ is a third-degree polynomial and so obtain a frequency equation

$$\nu^4[(221-45a^2)^2-104^2a^2]-2\nu^2(1-a^2)(1385-153a^2)+9(1-a^2)^2 = 0. \quad (4.22)$$

for the two cubic modes. For this and higher modes the algebra is extremely involved. When the paraboloid is circular, $a = 0$ and (4.22) gives $\nu = \frac{3}{13}$ and $\nu = \frac{1}{17}$ in agreement with Miles & Ball (1963) equation (3.16*b*), $s = 3, j = 2$ or $s = 1, j = 3$.

5. The effect of elliptic rotation on the simpler modes

When the liquid is rotating elliptically (see equations (4.7) and (4.8)), the vorticity is constant and the non-linear terms in equation (3.16) vanish.

Elliptic rotation is therefore an exact solution of the non-linear equations and we can consider perturbations from a state of elliptic rotation with vorticity Λ say (the exact solution for elliptic rotation when ϵ is not small has been considered elsewhere in Ball 1965). Accordingly we put

$$u = -\tfrac{1}{2}(1+a)y\Lambda - 2(1+a)y\psi - z\,\partial\psi/\partial y, \tag{5.1}$$

and

$$v = \tfrac{1}{2}(1-a)x\Lambda + 2(1-a)x\psi + z\,\partial\psi/\partial x, \tag{5.2}$$

in equation (3.16) and assume that ψ, which is now the perturbation from the basic value $\frac{1}{4}\Lambda$, is small. We then obtain

$$\begin{aligned} 4\frac{\partial\psi}{\partial t} + 3\left[(1-a)\,x\frac{\partial^2\psi}{\partial x\,\partial t} + (1+a)\,y\frac{\partial^2\psi}{\partial y\partial t}\right] + z\nabla^2\left(\frac{\partial\psi}{\partial t}\right) + (1-a)x\,[1+\tfrac{3}{2}\Lambda\,(3+a)]\frac{\partial\psi}{\partial y} \\ -(1+a)\,y\,[1+\tfrac{3}{2}\Lambda\,(3-a)]\frac{\partial\psi}{\partial x} + \tfrac{3}{2}\Lambda\,(1-a^2)xy\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) + \tfrac{3}{2}\Lambda[(1-a)^2x^2 \\ -(1+a)^2y^2]\frac{\partial^2\psi}{\partial x\,\partial y} + \tfrac{1}{2}z\Lambda[(1-a)\,x\frac{\partial}{\partial y}(\nabla^2\psi) - (1+a)\,y\frac{\partial}{\partial x}(\nabla^2\psi)] = 0. \end{aligned} \tag{5.3}$$

This equation, although a great deal more complicated than (4.6), still has polynomial solutions that 'correspond' to the polynomial solutions of (4.6).

We first determine the effect of the elliptic rotation on the linear mode by assuming, as before, that

$$\psi = \psi_{10}x + \psi_{01}y. \tag{5.4}$$

When we substitute this expression into equation (5.3) we obtain

$$\{(7-3a)\,d\psi_{10}/dt\} + (1-a)\,[1+\tfrac{3}{2}\Lambda(3+a)]\,\psi_{01} = 0, \tag{5.5}$$

$$\{(7+3a)\,d\psi_{01}/dt\} - (1+a)\,[1+\tfrac{3}{2}\Lambda(3-a)]\,\psi_{10} = 0. \tag{5.6}$$

We therefore have a motion with frequency ν given by

$$\nu^2 = (1-a^2)\,[1+\tfrac{3}{2}\Lambda(3+a)]\,[1+\tfrac{3}{2}\Lambda(3-a)]/(49-9a^2). \tag{5.7}$$

When ν^2 is positive the motion is oscillatory and substantially the same as before. There is, however, the interesting possibility of a negative ν^2, *implying instability for a certain range of negative* Λ, namely

$$-\tfrac{2}{3}/(3+a) > \Lambda > -\tfrac{2}{3}/(3-a). \tag{5.8}$$

This type of instability does not occur when the rotation is circular ($a = 0$) but no matter how small the ellipticity of the rotation there are always values of Λ that lead to instability.

Let us consider briefly the case where Λ is in the middle of the range of instability, i.e.

$$\Lambda = -\tfrac{2}{9}. \tag{5.9}$$

We put $\tau^2 = -\nu^{-2}$ so that

$$\tau^2 = 9(49-9a^2)/[a^2(1-a^2)]. \tag{5.10}$$

The solution for the unstable motion ($\tau > 0$) is given by

$$\psi = A\exp(t/\tau)\,[x\{(7+3a)(1-a)\}^{\frac{1}{2}}+y\{(7-2a)(1+a)\}^{\frac{1}{2}}], \tag{5.11}$$

which represents an additional circulation, roughly as depicted in figure 1 (*b*), the line dividing the circulation cells being inclined at 45° when the ellipse is nearly circular and at a greater angle to the major axis otherwise. The resultant streamlines, when this pattern is combined with the basic elliptic rotation, represent a single rotation cell in which the centre of rotation no longer coincides with the centre of the basin. This instability therefore represents a tendency for the centre of the circulation to migrate from the centre of the ellipse towards one 'end' of the ellipse when the ellipticity is large and into a quadrant in which xy is positive if the ellipticity is small. The minimum value of τ is about 40 so that this unstable motion develops very slowly.

We determine the effect of elliptic rotation on the 'quadratic' modes by putting

$$\psi = \psi_{00}+\psi_{20}x^2+2\psi_{11}xy+\psi_{02}y^2 \tag{5.12}$$

in equation (5.3) to obtain

$$\{(11-7a)\,d\psi_{20}/dt\}+\{(1-a)\,d\psi_{02}/dt\}+\{2(1-a)(1+6\Lambda)\psi_{11}\} = 0, \tag{5.13}$$

$$10d\psi_{11}/dt+(1-a)[1+3\Lambda(2+a)]\psi_{02}-(1+a)[1+3\Lambda(2-a)]\psi_{20} = 0, \tag{5.14}$$

$$\{(11+7a)\,d\psi_{02}/dt\}+\{(1+a)\,d\psi_{20}/dt\}-2(1+a)(1+6\Lambda)\psi_{11} = 0, \tag{5.15}$$

and

$$d(2\psi_{00}-\psi_{02}-\psi_{20})/dt = 0. \tag{5.16}$$

The frequency equation becomes

$$\nu\{5\nu^2(5-2a^2)-(1-a^2)(1+6\Lambda)(1+6\Lambda-2\Lambda a^2)\} = 0. \tag{5.17}$$

As before we have a zero-frequency mode and an oscillatory mode which may become exponentially unstable if Λ is negative and

$$-\tfrac{1}{6} > \Lambda > -1/(6-2a^2). \tag{5.18}$$

The function ψ gives a three-cell streamline pattern (as in § 4) which when added to the basic flow shows that the elliptic rotation cell may tend to elongate and ultimately break into two cells.

6. Necessary conditions for instability

(*a*) *A crude necessary condition independent of the ellipticity*

A crude necessary (but not sufficient) condition for the instability of elliptic rotation can be derived by considering the total angular momentum and energy of the motion. The angular momentum, in dimensionless form, is

$$J = \int h(vx-uy)\,dS, \tag{6.1}$$

where dS is an element of area, the range of integration covers the whole ellipse and, on our present approximation, we have

$$h = -z = 1-\tfrac{1}{2}(1-a)x^2-\tfrac{1}{2}(1+a)y^2. \tag{6.2}$$

Consider the identity

$$\partial(h^2v)/\partial x - \partial(h^2u)/\partial y + 2h(vx - uy) = h^2\zeta + 2ah(uy + vx), \tag{6.3}$$

where ζ is the vorticity. By integrating (6.3) over the whole ellipse, remembering that h is zero on the periphery, we obtain

$$J = \frac{1}{2}\int h^2\zeta\, dS + a\int h(uy + vx)\, dS. \tag{6.4}$$

The last integral can be written

$$\int h\frac{D(xy)}{Dt}\, dS = \frac{d}{dt}\int hxy\, dS, \tag{6.5}$$

which is zero, since the shape of the liquid is fixed, whence

$$J = \frac{1}{2}\int h^2\zeta\, dS. \tag{6.6}$$

Suppose the motion has been derived from a state of elliptic rotation by a process that conserves potential vorticity (as would, for instance, a spontaneous motion). If a liquid column of depth h, area dS and vorticity ζ, formerly had depth h_1, area dS_1 and vorticity Λ, then

$$h\, dS = h_1\, dS_1 \quad \text{(conservation of volume)}, \tag{6.7}$$

and $$(\zeta + 1)\, dS = (\Lambda + 1)\, dS_1 \quad \text{(conservation of potential vorticity)}, \tag{6.8}$$

and from (6.6) we can express the angular momentum in the form

$$\begin{aligned} J &= \frac{1}{2}\int h^2(\zeta + 1)\, dS - \frac{1}{2}\int h^2\, dS \\ &= \frac{1}{2}\int h^2(\Lambda + 1)\, dS_1 - \frac{1}{2}\int h^2\, dS. \end{aligned} \tag{6.9}$$

Now the shape of the liquid is unchanged so

$$\int h^2\, dS = \int h_1^2\, dS_1 = \int hh_1\, dS_1. \tag{6.10}$$

Furthermore, Λ is constant and equation (6.9) can be written

$$J = \tfrac{1}{2}(\Lambda + 1)\int h^2\, dS_1 - \frac{1}{2}\int h_1^2\, dS_1, \tag{6.11}$$

and if we denote the angular momentum of elliptic rotation with vorticity Λ by J_1 then

$$J - J_1 = \tfrac{1}{2}(\Lambda + 1)\int (h^2 - h_1^2)\, dS_1,$$

and using (6.10) $$J - J_1 = \tfrac{1}{2}(\Lambda + 1)\int (h - h_1)^2\, dS_1. \tag{6.12}$$

The integral in (6.12) is necessarily positive and this implies that any spontaneous breakdown of elliptic rotation will cause an *increase* in the total angular momentum if the absolute vorticity, $\Lambda + 1$, is *positive* and a decrease otherwise. It is shown in the following paragraphs that the absolute vorticity must be positive for the elliptic rotation to be unstable and such unstable motions must therefore be accompanied by an increase in angular momentum.

Let us now consider the kinetic energy of the liquid given by

$$K = \frac{1}{2}\int h(u^2+v^2)\,dS. \tag{6.13}$$

If we put
and

$$\left.\begin{aligned} u &= u_1+u' = -\tfrac{1}{2}(1+a)\,y\Lambda+u' \\ v &= v_1+v' = \tfrac{1}{2}(1-a)\,x\Lambda+v', \end{aligned}\right\} \tag{6.14}$$

(u_1, v_1) being the velocity field of elliptic rotation, then

$$K = \frac{1}{2}\int h\,[u_1^2+v_1^2+u'^2+v'^2+\Lambda\{(xv'-yu')-a(xv'+yu')\}]\,dS. \tag{6.15}$$

The first two terms in the integrand give the kinetic energy K_1 of the elliptic rotation and since the kinetic energy is constant we have

$$K = K_1.$$

The next two terms give the kinetic energy of the disturbance and are necessarily positive, the remaining terms are

$$\tfrac{1}{2}\Lambda\int h(xv'-yu')\,dS = \tfrac{1}{2}\Lambda(J-J_1),$$

and

$$\tfrac{1}{2}\Lambda a\int h(xv'+yu')\,dS = \tfrac{1}{2}\Lambda a\int h[uy+vx-\tfrac{1}{2}\Lambda(1-a)x^2+\tfrac{1}{2}\Lambda(1+a)y^2]\,dS,$$

where J_1 is the angular momentum of the elliptic rotation. The latter integral vanishes because of symmetry of the integrand and constancy of shape of the liquid (see (6.5)). Equation (6.15) now becomes

$$\int h(u'^2+v'^2)\,dS+\Lambda(J-J_1) = 0. \tag{6.16}$$

In the case of a circular basin there is no couple exerted on the liquid and J is constant, so we have

$$\int h(u'^2+v'^2)\,dS = 0,$$

which implies that $u' = v' = 0$ and the motion is *stable*. In the case of an elliptic basin we substitute for $J-J_1$, using (6.12), to obtain

$$\int h(u'^2+v'^2)\,dS+\tfrac{1}{2}\Lambda(\Lambda+1)\int (h-h_1)^2\,dS_1 = 0. \tag{6.17}$$

A spontaneous motion can only occur if the second term is negative, a *necessary condition for instability* is therefore

$$\Lambda(\Lambda+1) < 0, \quad \text{or} \quad -1 < \Lambda < 0. \tag{6.18}$$

The relative vorticity must be negative and the absolute vorticity positive for instability to occur and, as remarked previously, positive absolute vorticity implies an increase in angular momentum which in turn implies that an *unstable motion must exert a couple tending to oppose the rotation of the container.*

We can also set a limit to the rate of increase of J by the following transformations.

$$\int h(u'^2+v'^2)\,dS = \int h(u'+v')^2\,dS - 2\int hu'v'\,dS,$$

and
$$\int hu'v'\,dS = \int h(u+\tfrac{1}{2}(1+a)\Lambda y)\,(v-\tfrac{1}{2}(1-a)\Lambda x)\,dS$$
$$= \int h\{uv+\tfrac{1}{2}\Lambda[(1+a)yv-(1-a)xu]-\tfrac{1}{4}\Lambda^2(1-a^2)xy\}\,dS.$$

The term in Λ^2 vanishes because of the symmetry of h and the term in Λ can be written

$$\tfrac{1}{4}\Lambda\frac{d}{dt}\int h[(1+a)y^2-(1-a)x^2]\,dS = 0$$

because the liquid has constant shape, whence

$$\int h(u'^2+v'^2)\,dS = \int h(u'+v')^2\,dS - 2\int huv\,dS. \tag{6.19}$$

The last integral can be transformed further by putting

$$J = \int h(vx-uy)\,dS - a\int h(vx+uy)\,dS \tag{6.20}$$

(in this expression for J the second integral is identically zero, see (6.5)). Combining the two integrals we obtain

$$J = \int h[(1-a)\,vx-(1+a)\,uy]\,dS,$$

and differentiation with respect to time gives

$$\frac{dJ}{dt} = -\int h\{2auv + x(1-a)\,(\alpha\,\partial p/\partial y+u) - y(1+a)\,(\alpha\,\partial p/\partial x - v)\}\,dS$$
$$= -2a\int huv\,dS - \frac{1}{2}\frac{d}{dt}\int h[x^2(1-a)+y^2(1+a)]\,dS$$
$$+\int \alpha\left\{(1+a)\frac{\partial}{\partial x}(hpy)-(1-a)\frac{\partial}{\partial y}(hpx)\right\}dS.$$

The last integral vanishes because h is zero on the periphery and the second integral vanishes because the liquid has constant shape, and so we obtain the following simple expression of dJ/dt

$$dJ/dt = -2a\int huv\,dS. \tag{6.21}$$

From (6.16), (6.19) and (6.21) we obtain

$$a\int h(u'+v')^2\,dS + dJ/dt + a\Lambda(J-J_1) = 0$$

or
$$dJ/dt < -a\Lambda(J-J_1). \tag{6.22}$$

The time constant for exponentially increasing J is therefore not less than $-(a\Lambda)^{-1}$ (Λ is negative for instability, see (6.18)).

(*b*) *Necessary conditions for an almost circular basin*

Suppose the container (not necessarily elliptical) rotates with angular velocity ω_1 and the liquid rotates with angular velocity ω_2 relative to the container. In the limiting case, where the container is circular, we have a circular rotation with angular velocity $\omega_0 = \omega_1 + \omega_2$, a motion that we know to be stable. In the exponentially unstable modes the exponent must tend to zero as the container is continuously deformed into a circular paraboloid and, provided the frequency is measured relative to axes rotating with angular velocity ω_1, will ultimately correspond to a mode of zero frequency. It is easy to determine the circumstances in which the various modes have zero frequency since we know the complete solution in the case of the circular paraboloid (see Miles & Ball 1963). This analysis will then give *necessary* conditions for pure exponential instability for any small continuous distortion from the circular shape. Different considerations apply to oscillatory unstable (overstable) motions about which a little is said subsequently.

The frequency of the (s,j) second-class mode for a liquid rotating with angular velocity ω_0 in a circular paraboloid, when ϵ is small, is given by

$$\nu = sk\omega_0, \tag{6.23}$$

where $k = 2/[s+2(j+s)(j-1)]$ (s and j are integers, $s \geqslant 1, j \geqslant 2$), (6.24)

and the frequency is measured relative to the rotating liquid (see Miles & Ball 1963, equations (3.16*b*) and (3.17)). A positive frequency here indicates a mode that rotates in a positive direction and s is the 'angular wave-number' so the angular velocity of the mode, relative to the liquid, is

$$\omega_{s,j} = \nu/s = k\omega_0. \tag{6.25}$$

The 'absolute' angular velocity of the mode, i.e. relative to stationary co-ordinates, is

$$\omega_{s,j} + \omega_0 = \omega_0(1+k). \tag{6.26}$$

If we take co-ordinates rotating with angular velocity ω_1 where

$$\omega_0 = \omega_1 + \omega_2, \tag{6.27}$$

then, relative to these new co-ordinates, ω_2 is the angular velocity of the liquid and the angular velocity of the mode is

$$\omega_{s,j} + \omega_0 - \omega_1 = \omega_1 k + \omega_2(1+k). \tag{6.28}$$

We now choose ω_1 so that the angular velocity (consequently also the frequency) of the mode is zero, then

$$\omega_2/\omega_1 = -k/(1+k). \tag{6.29}$$

In terms of the dimensionless quantity Λ, used previously, and substituting for k from (6.24), we obtain

$$\Lambda = \omega_2/\omega_1 = -2/[2+s+2(j+s)(j-1)]. \tag{6.30}$$

If a circular paraboloid, rotating with angular velocity ω_1, is slightly deformed and the liquid is rotating relative to the container with angular velocity ω_2, then (6.30) *gives the necessary condition for the* (s, j) *mode to be exponentially unstable.* This result is in agreement with our previous calculations for elliptic deformation since we obtain

$$\Lambda = -\tfrac{2}{9} \quad (s = 1, j = 2;\ \text{compare (5.8) when } a \to 0),$$

and
$$\Lambda = -\tfrac{1}{6} \quad (s = 2, j = 2;\ \text{compare (5.18) when } a \to 0).$$

The two cubic modes give

$$\Lambda = -\tfrac{2}{19} \ (s = 1, j = 3), \quad \Lambda = -\tfrac{2}{15} \ (s = 3, j = 2). \tag{6.31}$$

Necessary conditions for the occurrence of oscillatory unstable motions are rather more complicated and we confine our attention to a slight elliptic deformation of the paraboloid. The nth-degree modes in an elliptic paraboloid satisfy an $(n+1)$th-degree frequency equation derived from the $n+1$ differential equations for the $n+1$ coefficients of the nth-degree terms in the polynomial for ψ. If n is *odd* this equation is in ν^2, if n is *even* there is always one zero root, corresponding to motion along the contours, and the remaining part of the equation is in ν^2. When the ellipse is deformed into a circle, the roots of this equation give modes that correspond to those for which

$$s + 2j - 4 = n,$$

and all of these modes are stable. If we suppose that the container is rotating at an angular velocity whose value is so selected that two of these roots are equal, then slight deformation *may* lead to complex roots some of which would correspond to oscillatory unstable modes.

It is immediately apparent that neither the linear nor the quadratic modes can be unstable in this way since there is only the one oscillatory mode in each of these groups. There are, however, two oscillatory cubic modes corresponding to

$$n = 3, s = 1, j = 3, k = \tfrac{2}{17}; \quad n = 3, s = 3, j = 2, k = \tfrac{2}{13}.$$

For instability ω_1 and ω_2 must be selected so that the frequencies, given by

$$\nu = s(\omega_{s,j} + \omega_0 - \omega_1) = s[\omega_1 k + \omega_2(1+k)], \quad \text{(see (6.25) and (6.28))}$$

have the same magnitude. This leads to

$$\Lambda = \omega_2/\omega_1 = -\tfrac{32}{253} \text{ or } -\tfrac{38}{259},$$

which together with (6.31) give four *possible* values of Λ for instability.

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